TABLOIDS AND WEIGHTED SUMS OF CHARACTERS OF CERTAIN MODULES OF THE SYMMETRIC GROUPS

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ABSTRACT. We consider certain modules of the symmetric groups whose basis elements are called tabloids. Some of these modules are isomorphic to subspaces of the cohomology rings of subvarieties of flag varieties as modules of the symmetric groups. We give a combinatorial description for some weighted sums of their characters, i.e., we introduce combinatorial objects called $(\rho, \boldsymbol{l})$-tabloids and rewrite weighted sums of characters as the numbers of these combinatorial objects. We also consider the meaning of these combinatorial objects, i.e., we construct a correspondence between $(\rho, \boldsymbol{l})$-tabloids and tabloids whose images are eigenvectors of the action of an element of cycle type ρ in quotient modules.

1. INTRODUCTION

Let W be a finite reflection group. In some $\mathbb{Z}$-graded W-modules $R = \bigoplus_d R^d$, we have a phenomenon called "coincidence of dimensions" ([4], [3] and so on), i.e., some integers l satisfy the equations

$$\dim \bigoplus_{i \in \mathbb{Z}} R^{il+k} = \dim \bigoplus_{i \in \mathbb{Z}} R^{il+k'}$$

for all k and k'. Induced modules often give a proof of the phenomenon. More specifically, let a subgroup $H(l)$ of W and $H(l)$-modules $z(k; l)$ satisfy

$$\bigoplus_{i \in \mathbb{Z}} R^{il+k} \simeq \mathrm{Ind}_{H(l)}^{W} z(k; l), \qquad \dim z(k; l) = \dim z(k'; l)$$

for all k and k', where $\mathrm{Ind}_{H(l)}^{W} z(k; l)$ denotes the induced module. Since

$$\dim \mathrm{Ind}_{H(l)}^{W} z(k; l) = |W/H(l)| \cdot \dim z(k; l),$$

we can prove the phenomenon by the datum $(H(l), \{\, z(k; l) \,\})$.

We consider the case where W is the m-th symmetric group S_m and R are the S_m-modules R_μ called Springer modules. The Springer modules R_μ are graded algebras parametrized by partitions $\mu \vdash m$. As S_m-modules, R_μ are isomorphic to cohomology rings of the variety of the flags fixed by a unipotent matrix the sizes of which Jordan blocks are μ. (See [2, 7, 8]. See also [1, 9] for algebraic construction.) Let μ be an l-partition, where an l-partition means a partition whose multiplicities are divisible by l. To prove coincidence of dimensions of the Springer module R_μ by induced modules, Morita-Nakajima [5] explicitly calculated the Green polynomials corresponding to μ at l-th roots of unity. These values are nonnegative integers.

Our first motivation for this paper is to describe these nonnegative values of the Green polynomials as numbers of some combinatorial objects. Our second motivation is to give a meaning of the combinatorial objects in terms of modules $\mathrm{Ind}_{H_\mu(l)}^{S_m} Z_\mu(k; l)$ in Morita-Nakajima [5]. For these purposes, we introduce some

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S_m-modules, which are realizations of $\mathrm{Ind}_{H_\mu(l)}^{S_m} Z_\mu(k; l)$ for special parameters, and give a combinatorial description for weighted sums of their characters.

In Section 2, we introduce S_m-modules $M^{\boldsymbol{\mu}}$ and their quotient modules $M^{\boldsymbol{\mu}}(k; \boldsymbol{l})$ for some n-tuple $\boldsymbol{\mu}$ of Young diagrams. When $n = 1$, this module $M^{(\mu)}(k; (l))$ is a realization of $\mathrm{Ind}_{H_\mu(l)}^{S_m} Z_\mu(k; l)$ in [5]. We also introduce combinatorial objects called marked $(\rho, \boldsymbol{l})$-tabloids to describe weighted sums of characters of $M^{\boldsymbol{\mu}}(k; \boldsymbol{l})$. When $n = 1$, the number of marked $(\rho, (l))$-tabloids coincides with the right hand side of the explicit formula (3.1) of Green polynomials in [5]. Our main result is the description of a weighted sum

$$\sum_{k \in \mathbb{Z}/l\mathbb{Z}} \zeta_l^{jk} \, \mathrm{Char}\left(M^{\boldsymbol{\mu}}(k; \boldsymbol{l})\right)(\sigma)$$

of characters of $M^{\boldsymbol{\mu}}(k; \boldsymbol{l})$ as the number of marked $(\rho, \boldsymbol{\gamma})$-tabloids on $\boldsymbol{\mu}$ for the primitive l-th root ζ_l of unity and $\sigma \in S_m$ of cycle type ρ in Section 3. We prove the main result in Section 4 by constructing bijections.

2. Notation and Definition

We identify a partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots)$ of m with its Young diagram $\left\{ (i, j) \in \mathbb{N}^2 \,\middle|\, 1 \leq j \leq \mu_i \right\}$ with m boxes. If μ is a Young diagram with m boxes, we write $\mu \vdash m$ and identify a Young diagram μ with the array of m boxes having left-justified rows with the i-th row containing μ_i boxes; for example,

$$(2, 2, 1) = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \multicolumn{1}{|c|}{} \\ \cline{1-1} \end{array} \vdash 5.$$

For an integer l, a Young diagram μ is called an *l-partition* if multiplicities $m_i = |\{ k \,|\, \mu_k = i \}|$ of i are divisible by l for all i.

Let μ be a Young diagram with m boxes. We call a map T a *numbering on μ with $\{1, \ldots, n\}$* if T is an injection $\mu \ni (i, j) \mapsto T_{i,j} \in \{1, \ldots, n\}$. We identify a map $T : \mu \to \mathbb{N}$ with a diagram putting $T_{i,j}$ in each box in the (i, j) position; for example,

$$\begin{array}{|c|c|} \hline 2 & 3 \\ \hline 6 & 1 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

is identified with a numbering T on $(2, 2, 1)$ which maps $T_{1,1} = 2$, $T_{1,2} = 3$, $T_{2,1} = 6$, $T_{2,2} = 1$, $T_{3,1} = 5$. For $\mu \vdash m$, t_μ denotes the numbering which maps $(t_\mu)_{i,j} = j + \sum_{k=0}^{i-1} \mu_k$; i.e., the numbering obtained by putting numbers from 1 to m on boxes of μ from left to right in each row, starting in the top row and moving to the bottom row. For example,

$$t \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \multicolumn{1}{|c|}{} \\ \cline{1-1} \end{array} = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

Two numberings T and T' on $\mu \vdash m$ are said to be *row-equivalent* if their corresponding rows consist of the same elements. We call a row-equivalence class $\{T\}$ a *tabloid*.

Let T be a numbering T on a Young diagram $\mu \vdash m$ with $\{1, \ldots, n\}$. Then $\sigma \in S_n$ acts on T from left as $(\sigma T)_{i,j} = \sigma(T_{i,j})$. For example,

$$(1, 2, 3, 4) \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

This left action induces a left action on tabloids by $\sigma\{T\} = \{\sigma T\}$.

For a numbering T on $\mu \vdash m$ with $\{1,\ldots,n\}$, we define S_T to be the subgroup

$$S_{\left\{\, T_{1,1},T_{1,2},\ldots,T_{1,\mu_1}\,\right\}} \times S_{\left\{\, T_{2,1},T_{2,2},\ldots,T_{2,\mu_2}\,\right\}} \times \cdots$$

of the n-th symmetric group S_n, where $S_{\{\,i_1,\ldots,i_k\,\}}$ denotes the symmetric group of the letters $\{\,i_1,\ldots,i_k\,\}$. It is obvious that S_T and the Young subgroup S_μ are isomorphic as groups for a numbering T on $\mu \vdash m$. It is also clear that $\sigma\{T\} = \{T\}$ for $\sigma \in S_T$.

For a numbering T on an l-partition $\mu \vdash m$, we define $a_{T,l}$ to be the product

$$\prod_{(li+1,j)\in\mu} (T_{li+1,j},T_{li+2,j},\ldots,T_{li+l,j})$$

of m/l cyclic permutations of length l. For example, $a_{t_{(2,2,1,1)},2} = (13)(24)(56)$. We write $a_{\mu,l}$ for $a_{t_\mu,l}$.

Let μ be an l-partition of m and $\langle a_{\mu,l}\rangle$ the cyclic group of order l generated by $a_{\mu,l}$. For each numbering T on μ with $\{1,\ldots,n\}$, there exists $\tau_T \in S_n$ such that $T = \tau_T t_\mu$. Since the map $\tau_T|_{\{\,1,\ldots,m\,\}}$ restricting τ_T to $\{\,1,\ldots,m\,\}$ is unique, $\sigma \in \langle a_{\mu,l}\rangle$ acts on T from right as $T\sigma = \tau_T \sigma t_\mu$. For each numbering T on an l-partition μ, the $(\overline{r} + lq)$-th row of $Ta_{\mu,l}$ is the $(\overline{r+1} + lq)$-th row of T, where $\overline{r}$ and $\overline{r+1}$ are in $\mathbb{Z}/l\mathbb{Z} = \{\,1,\ldots,l\,\}$. This right action also induces a right action on tabloids by $\{T\}\sigma = \{T\sigma\}$.

In this paper, we consider n-tuples of Young diagrams. Throughout this paper, let $\boldsymbol{m} = (m_1,m_2,\ldots,m_n)$ and $\boldsymbol{l} = (l_1,l_2,\ldots,l_n)$ be n-tuples of positive integers, m the sum $\sum_h m_h$, l the least common multiple of $\{\,l_i\,\}$ and ζ_k the primitive k-th root of unity. We call an n-tuple $\boldsymbol{\mu} = (\mu^{(1)},\mu^{(2)},\ldots,\mu^{(n)})$ of Young diagrams an $\boldsymbol{l}$-partition of $\boldsymbol{m}$ if $\mu^{(h)}$ is an l_h-partition of m_h for each h. We identify an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ with the disjoint union $\coprod_h \mu^{(h)} = \{\,(i,j;h)\,\big|\,(i,j)\in\mu^{(h)}\,\}$ of Young diagrams $\mu^{(h)}$. We call an n-tuple $\boldsymbol{T} = (T^{(1)},\ldots,T^{(n)})$ of numberings $T^{(h)}$ on $\mu^{(h)}$ a numbering on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ if the map $\boldsymbol{T}: \boldsymbol{\mu} \ni (i,j;h) \mapsto T_{i,j}^{(h)} \in \{\,1,\ldots,m\,\}$ is bijective. For an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ of $\boldsymbol{m}$, $\boldsymbol{t_\mu}$ denotes the n-tuple of the numberings $t_{\boldsymbol{\mu}}^{(h)}$ which maps $(i,j;h)$ to $(t_{\boldsymbol{\mu}}^{(h)})_{i,j} = (t_{\mu^{(h)}})_{i,j} + \sum_{k=1}^{h-1} m_k$; i.e., $\boldsymbol{t_\mu}$ is the numbering on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ obtained by putting numbers from 1 to m on boxes of $\boldsymbol{\mu}$ from left to right in each row, starting in the top row and moving to the bottom in each Young diagram, starting from $\mu^{(1)}$ to $\mu^{(n)}$. For example,

$$\boldsymbol{t}\left(\begin{array}{cc}\ ,\ \end{array}\right) = \left(\begin{array}{cc}\boxed{\begin{array}{cc}1&2\\3&4\\5&\\6&\end{array}} &,& \boxed{\begin{array}{cc}7&8\\9&\end{array}}\end{array}\right).$$

Let $\boldsymbol{T}$ be a numbering on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ of $\boldsymbol{m}$. We define $S_{\boldsymbol{T}}$ to be the subgroup $S_{\boldsymbol{T}} = S_{T^{(1)}} \times S_{T^{(2)}} \times \cdots \times S_{T^{(n)}}$ of S_m. The subgroup $S_{\boldsymbol{T}}$ and the Young subgroup $S_{\overline{\boldsymbol{\mu}}}$ are isomorphic as groups, where $\overline{\boldsymbol{\mu}}$ is the partition obtained from $(\mu_1^{(1)},\mu_1^{(2)},\ldots,\mu_1^{(n)},\mu_2^{(1)},\mu_2^{(2)},\ldots,\mu_2^{(n)},\ldots)$ by sorting in descending order. We write $S_{\boldsymbol{\mu}}$ for $S_{\boldsymbol{t_\mu}}$. We define $a_{\boldsymbol{T},\boldsymbol{l}}$ to be $a_{T^{(1)},l_1} \cdot a_{T^{(2)},l_2} \cdots a_{T^{(n)},l_n}$. We write $a_{\boldsymbol{\mu},\boldsymbol{l}}$ for $a_{\boldsymbol{t_\mu},\boldsymbol{l}}$

Two numberings $\boldsymbol{T}$ and $\boldsymbol{S}$ on an $\boldsymbol{l}$-partition of $\boldsymbol{m}$ are said to be *row-equivalent* if $T^{(h)}$ and $S^{(h)}$ are row-equivalent for each h. The set of numberings whose components are arranged in ascending order in each row is a complete set of representatives for row-equivalence classes. A row-equivalence class of a numbering $\boldsymbol{T}$ on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ is an n-tuple $(\{T^{(1)}\},\{T^{(2)}\},\ldots,\{T^{(n)}\})$ of tabloids $\{T^{(h)}\}$ on $\mu^{(h)}$. We also call a row-equivalence class $(\{T^{(h)}\})$ of numbering $\boldsymbol{T}$ on an $\boldsymbol{l}$-partition a *tabloid on an $\boldsymbol{l}$-partition*. We also write $\{\boldsymbol{T}\}$ for $(\{T^{(h)}\})$. The set of all tabloids

on an l-partition $\boldsymbol{\mu}$ of $\boldsymbol{m}$ is denoted by $\mathbb{T}_{\boldsymbol{\mu}}$. We define $M^{\boldsymbol{\mu}}$ to be the $\mathbb{C}$-vector space $\mathbb{C}\mathbb{T}_{\boldsymbol{\mu}}$ whose basis is the set $\mathbb{T}_{\boldsymbol{\mu}}$ of tabloids on $\boldsymbol{\mu}$.

Let $\boldsymbol{T}$ be a numbering on an l-partition of $\boldsymbol{m}$. Then $\sigma \in S_m$ acts on $\boldsymbol{T}$ from left as $\sigma(T^{(h)}) = (\sigma T^{(h)})$. This left action induces a left action on tabloids by $\sigma\{\boldsymbol{T}\} = \{\sigma\boldsymbol{T}\}$. For the partition $\overline{\boldsymbol{\mu}} \vdash m$, $M^{\boldsymbol{\mu}}$ and $\mathrm{Ind}_{S_{\overline{\boldsymbol{\mu}}}}^{S_m} 1$ are isomorphic as left S_m-modules, where 1 denotes the the trivial module of the Young subgroup $S_{\overline{\boldsymbol{\mu}}}$.

Let $\boldsymbol{T}$ be a numbering on an l-partition $\boldsymbol{\mu}$ of $\boldsymbol{m}$. Since there uniquely exists $\tau_{\boldsymbol{T}} \in S_m$ such that $\boldsymbol{T} = \tau_{\boldsymbol{T}} \boldsymbol{t}_{\boldsymbol{\mu}}$, $\sigma \in \langle a_{\boldsymbol{\mu},l}\rangle$ acts on $\boldsymbol{T}$ from right as $\boldsymbol{T}\sigma = \tau_{\boldsymbol{T}}\sigma\boldsymbol{t}_{\boldsymbol{\mu}}$. This right action also induces a right action on tabloids by $\{\boldsymbol{T}\}\sigma = \{\boldsymbol{T}\sigma\}$.

Next we introduce S_m-modules $M^{\boldsymbol{\mu}}(k;\boldsymbol{l})$, one of main objects in this paper. We need some definitions to introduce $M^{\boldsymbol{\mu}}(k;\boldsymbol{l})$.

Definition 2.1. Let $\mathbb{T}_{\boldsymbol{\mu}}^l$ be the subset $\left\{ a_{\boldsymbol{\mu},l}^i\{\boldsymbol{t}_{\boldsymbol{\mu}}\} \,\middle|\, i \in \mathbb{Z}/l\mathbb{Z} \right\}$ of tabloids for an l-partition $\boldsymbol{\mu}$ of $\boldsymbol{m}$. We define $Z_{\boldsymbol{\mu}}(\boldsymbol{l})$ to be the $\mathbb{C}$-vector space $\mathbb{C}\mathbb{T}_{\boldsymbol{\mu}}^l$ whose basis is $\mathbb{T}_{\boldsymbol{\mu}}^l$. This l-dimensional vector space is a left module of the semi-direct product $S_{\boldsymbol{\mu}} \rtimes \langle a_{\boldsymbol{\mu},l}\rangle$ and a right module of the cyclic group $\langle a_{\boldsymbol{\mu},l}\rangle$ of order l.

For $k \in \mathbb{Z}/l\mathbb{Z}$, let $I_{\boldsymbol{\mu}}(k;\boldsymbol{l})$ denote the submodule of $Z_{\boldsymbol{\mu}}(\boldsymbol{l})$ generated by

$$\left\{ a_{\boldsymbol{\mu},l}^i\{\boldsymbol{t}_{\boldsymbol{\mu}}\} - \zeta_l^{ki}\{\boldsymbol{t}_{\boldsymbol{\mu}}\} \,\middle|\, i \in \mathbb{Z}/l\mathbb{Z} \right\}.$$

We define $Z_{\boldsymbol{\mu}}(k;\boldsymbol{l})$ to be the quotient module

$$Z_{\boldsymbol{\mu}}(\boldsymbol{l})/I_{\boldsymbol{\mu}}(k;\boldsymbol{l}).$$

For each k, $Z_{\boldsymbol{\mu}}(k;\boldsymbol{l})$ is a one-dimensional left module of the semi-direct product $S_{\boldsymbol{\mu}} \rtimes \langle a_{\boldsymbol{\mu},l}\rangle$. This left $S_{\boldsymbol{\mu}} \rtimes \langle a_{\boldsymbol{\mu},l}\rangle$-module $Z_{\boldsymbol{\mu}}(k;\boldsymbol{l})$ is generated by $\{\boldsymbol{t}_{\boldsymbol{\mu}}\}$, and $a_{\boldsymbol{\mu},l}$ acts on $\{\boldsymbol{t}_{\boldsymbol{\mu}}\}$ by

$$a_{\boldsymbol{\mu},l}\{\boldsymbol{t}_{\boldsymbol{\mu}}\} = \zeta_l^k\{\boldsymbol{t}_{\boldsymbol{\mu}}\}$$

in $Z_{\boldsymbol{\mu}}(\boldsymbol{l})/I_{\boldsymbol{\mu}}(k;\boldsymbol{l})$.

Let $\widetilde{I}_{\boldsymbol{\mu}}(k;\boldsymbol{l})$ be $\mathbb{C}[S_m]I_{\boldsymbol{\mu}}(k;\boldsymbol{l})$. Finally, we define an S_m-module $M^{\boldsymbol{\mu}}(k;\boldsymbol{l})$ to be

$$M^{\boldsymbol{\mu}}/\widetilde{I}_{\boldsymbol{\mu}}(k;\boldsymbol{l}).$$

By definition, the S_n-module $M^{\boldsymbol{\mu}}(k;\boldsymbol{l})$ is a realization of the induced module $\mathrm{Ind}_{S_{\boldsymbol{\mu}} \rtimes \langle a_{\boldsymbol{\mu},l}\rangle}^{S_m} Z_{\boldsymbol{\mu}}(k;\boldsymbol{l})$.

Remark 2.2. For an l-partition μ of m, our module $M^{(\mu)}(k;(l))$ gives a realization of the S_m-module $\mathrm{Ind}_{H_\mu(l)}^{S_m} Z_\mu(k;l)$ in Morita-Nakajima [5]. For n-tuple $\{\,l_h\,\}$ of integers, $M^{\boldsymbol{\mu}}(k;\boldsymbol{l})$ is a realization of the induced module

$$\mathrm{Ind}_{S_{m_1}\times\cdots\times S_{m_n}}^{S_{m_1+\cdots+m_n}} M^{\mu^{(1)}}(k;l_1) \otimes \cdots \otimes M^{\mu^{(n)}}(k;l_n),$$

where $M^\mu(k;l)$ denotes $M^{(\mu)}(k;(l))$.

Remark 2.3. Since $\widetilde{I}_{\boldsymbol{\mu}}(k;\boldsymbol{l}) = \mathbb{C}[S_m]I_{\boldsymbol{\mu}}(k;\boldsymbol{l})$ is generated by

$$\left\{ \tau a_{\boldsymbol{\mu},l}^i\{\boldsymbol{t}_{\boldsymbol{\mu}}\} - \zeta_l^{ik}\tau\{\boldsymbol{t}_{\boldsymbol{\mu}}\} \,\middle|\, i \in \mathbb{Z}/l\mathbb{Z}, \tau \in S_m \right\},$$

$\widetilde{I}_{\boldsymbol{\mu}}(k;\boldsymbol{l})$ is also generated by

$$\left\{ \{\boldsymbol{T}\}a_{\boldsymbol{\mu},l}^i - \zeta_l^{ik}\{\boldsymbol{T}\} \,\middle|\, \{\boldsymbol{T}\} \in \mathbb{T}_{\boldsymbol{\mu}}, i \in \mathbb{Z}/l\mathbb{Z} \right\}.$$

Hence $a_{\boldsymbol{\mu},l}$ acts on tabloids $\{\boldsymbol{T}\}$ by

$$\{\boldsymbol{T}\}a_{\boldsymbol{\mu},l} = \zeta_l^k\{\boldsymbol{T}\}$$

in $M^{\boldsymbol{\mu}}(k;\boldsymbol{l})$.

We introduce the following combinatorial objects to describe the characters of $M^{\boldsymbol{\mu}}(k;\boldsymbol{l})$.

Definition 2.4. For a Young diagram $\rho \vdash m$, we call a map $Y : \boldsymbol{\mu} \to \mathbb{N}$ a $(\rho, \boldsymbol{l})$-*tabloid* on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ of $\boldsymbol{m}$ if the following are satisfied:

- $\left| Y^{-1}(\{\, k \,\}) \right| = \rho_k$ for all k,
- for each k, there exist $h \in \mathbb{N}$ and $(i', j') \in \mathbb{N}^2$ such that ρ_k is divisible by l_h and

$$Y^{-1}(\{\, k \,\}) = \left\{ (i + i', j + j'; h) \,\middle|\, (i, j) \in \left(\left(\frac{\rho_k}{l_h} \right)^{l_h} \right) \vdash \rho_k \right\},$$

- for each $(i, j; h)$, $(i, k; h) \in \boldsymbol{\mu}$, $Y(i, j; h) \leq Y(i, k; h)$ if $j \leq k$.

Example 2.5. For example,

$$\left(\begin{array}{|c|c|} \hline 3 & 4 \\ \hline 3 & 4 \\ \hline 1 & 1 \\ \hline 1 & 1 \\ \hline \end{array} \; , \; \begin{array}{|c|c|c|} \hline 2 & 2 & 5 \\ \hline \end{array} \right)$$

is a $((4, 2, 2, 2, 1), (2, 1))$-tabloid on $((2, 2, 2, 2), (3))$.

Definition 2.6. We call a pair (Y, c) a *marked* $(\rho, \boldsymbol{l})$-*tabloid* on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ of $\boldsymbol{m}$ if the following are satisfied:

- Y is a $(\rho, \boldsymbol{l})$-tabloid on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$,
- c is a map from $\{\, i \,|\, \rho_i \neq 0 \,\}$ to $\coprod_h \mathbb{Z}/l_h\mathbb{Z}$,
- $c(i)$ is in $\mathbb{Z}/l_h\mathbb{Z}$ if $Y^{-1}(\{\, i \,\}) \subset \mu^{(h)}$.

For a marked $(\rho, \boldsymbol{l})$-tabloid (Y, c), the inverse image $Y^{-1}(\{\, i \,\})$ has l_h rows and $c(i)$ is in $\mathbb{Z}/l_h\mathbb{Z} = \{\, 1, \ldots, l_h \,\}$ if $Y^{-1}(\{\, i \,\})$ is in $\mu^{(h)}$. We identify (Y, c) with the diagram obtained from the diagram of Y by putting $*$ in the left-most box of the $c(i)$-th row of the inverse image $Y^{-1}(\{\, i \,\})$, where we identify $\mathbb{Z}/l_h\mathbb{Z}$ with the set $\{\, 1, \ldots, l_h \,\}$ of complete representatives.

Example 2.7. Let

$$Y = \left(\begin{array}{|c|c|} \hline 3 & 4 \\ \hline 3 & 4 \\ \hline 1 & 1 \\ \hline 1 & 1 \\ \hline \end{array} \; , \; \begin{array}{|c|c|c|} \hline 2 & 2 & 5 \\ \hline \end{array} \right)$$

and let c be the map such that $c(1) = 2$, $c(3) = 1$, $c(4) = 2 \in \mathbb{Z}/2\mathbb{Z}$ and $c(2) = c(5) = 1 \in \mathbb{Z}/1\mathbb{Z}$, then (Y, c) is a marked $(2, 1)$-tabloid. We write

$$\left(\begin{array}{|c|c|} \hline 3^* & 4 \\ \hline 3 & 4^* \\ \hline 1 & 1 \\ \hline 1^* & 1 \\ \hline \end{array} \; , \; \begin{array}{|c|c|c|} \hline 2^* & 2 & 5^* \\ \hline \end{array} \right)$$

for (Y, c).

Remark 2.8. It follows from a direct calculation that the number of marked $(\rho, (l))$-tabloids on an (l)-partition (μ) equals the right hand side of the equation (3.1) in Morita-Nakajima [5].

Definition 2.9. Let $\boldsymbol{\mu}$ be an $\boldsymbol{l}$-partition of $\boldsymbol{m}$ and $\boldsymbol{\gamma} = (\gamma_h)$ an n-tuple of integers such that l_h is divisible by γ_h. For a Young diagram $\rho \vdash m$, we call a map $Y : \boldsymbol{\mu} \to \mathbb{N}$ a $(\rho, \boldsymbol{\gamma}, \boldsymbol{l})$-*tabloid* on $\boldsymbol{\mu}$ if the following are satisfied:

- $\left| Y^{-1}(\{\, k \,\}) \right| = \rho_k$ for all k,
- for each k, there exist h and $(i', j') \in \mathbb{N}^2$ such that ρ_k is divisible by γ_h and

$$Y^{-1}(\{\, k \,\}) = \left\{ \left(\frac{i l_h}{\gamma_h} + i', j + j'; h \right) \,\middle|\, (i, j) \in \left(\left(\frac{\rho_k}{\gamma_h} \right)^{\gamma_h} \right) \vdash \rho_k \right\},$$

- for each $(i,j;h), (i,k;h) \in \boldsymbol{\mu}$, $Y(i,j;h) \le Y(i,k;h)$ if $j \le k$.

Example 2.10. For example,

$$
\left(
\begin{array}{|c|c|c|c|}
\hline
2 & 2 & 2 & 4 \\
\hline
1 & 1 & 1 & 1 \\
\hline
2 & 2 & 2 & 4 \\
\hline
1 & 1 & 1 & 1 \\
\hline
\end{array}
\;,\;
\begin{array}{|c|c|}
\hline
3 & 3 \\
\hline
5 & 5 \\
\hline
6 \\
\cline{1-1}
\end{array}
\right)
$$

is an $((8,6,2,2,2,1),(2,1),(4,1))$-tabloid on $((4,4,4,4),(5))$.

A $(\rho,\boldsymbol{l},\boldsymbol{l})$-tabloid on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ is a $(\rho,\boldsymbol{l})$-tabloid on $\boldsymbol{\mu}$.

For an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ and an n-tuple $\boldsymbol{\gamma} = (\gamma_h)$ such that l_h is divisible by γ_h, it follows that

$$|\{\, Y \,|\, \text{a } (\rho,\boldsymbol{\gamma},\boldsymbol{l})\text{-tabloid on } \boldsymbol{\mu} \,\}| = |\{\, Y \,|\, \text{a } (\rho,\boldsymbol{\gamma})\text{-tabloid on } \boldsymbol{\mu} \,\}|.$$

Definition 2.11. We call a pair (Y,c) a *marked $(\rho,\boldsymbol{\gamma},\boldsymbol{l})$-tabloid* on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ of $\boldsymbol{m}$ the following are satisfied:

- Y is a $(\rho,\boldsymbol{\gamma},\boldsymbol{l})$-tabloid on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$,
- c is a map from $\{\, i \,|\, \rho_i \ne 0 \,\}$ to $\coprod_h \mathbb{Z}/\gamma_h\mathbb{Z}$,
- $c(i)$ is in $\mathbb{Z}/\gamma_h\mathbb{Z}$ if $Y^{-1}(\{\, i \,\}) \subset \mu^{(h)}$.

Similar to the case of marked $(\rho,\boldsymbol{l})$-tabloids, we identify a marked $(\rho,\boldsymbol{\gamma},\boldsymbol{l})$-tabloid (Y,c) with the diagram obtained from the diagram of Y by putting $*$ in the left-most box of the $c(i)$-th row of the inverse image $Y^{-1}(\{\, i \,\})$.

Example 2.12. For example,

$$
\left(
\begin{array}{|c|c|}
\hline
3^* & 4 \\
\hline
1 & 1 \\
\hline
3 & 4^* \\
\hline
1^* & 1 \\
\hline
\end{array}
\;,\;
\begin{array}{|c|c|c|}
\hline
2^* & 2 & 5^* \\
\hline
\end{array}
\right)
$$

is a marked $((4,2,2,2,1),(2,1),(4,1))$-tabloid.

For an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ and an n-tuple $\boldsymbol{\gamma} = (\gamma_h)$ such that l_h is divisible by γ_h, it follows that

$$
(1) \qquad
\begin{aligned}
&|\{\, (Y,c) \,|\, \text{a marked } (\rho,\boldsymbol{\gamma},\boldsymbol{l})\text{-tabloid on } \boldsymbol{\mu} \,\}| \\
&= |\{\, (Y,c) \,|\, \text{a marked } (\rho,\boldsymbol{\gamma})\text{-tabloid on } \boldsymbol{\mu} \,\}|.
\end{aligned}
$$

3. Main Results

The following are the main results of this paper, proved in Section 4.

Theorem 3.1. *For an integer j, let $\boldsymbol{\mu}$ be an $\boldsymbol{l}$-partition and $\boldsymbol{\gamma}$ an n-tuple of integers such that γ_h is the order of $\zeta_{l_h}^j$. For $\sigma \in S_m$ of cycle type ρ,*

$$\sum_{k \in \mathbb{Z}/l\mathbb{Z}} \zeta_l^{jk} \, \mathrm{Char}\left(M^{\boldsymbol{\mu}}(k;\boldsymbol{l})\right)(\sigma) = |\{\, (Y,c) \,|\, \text{a marked } (\rho,\boldsymbol{\gamma})\text{-tabloid on } \boldsymbol{\mu} \,\}|.$$

Theorem 3.2. *For an integer j, let $\boldsymbol{\mu}$ be an $\boldsymbol{l}$-partition and $\boldsymbol{\gamma}$ an n-tuple of integers such that γ_h is the order of $a_{\mu^{(h)},l_h}^j$. Tabloids $\boldsymbol{T}$ on $\boldsymbol{\mu}$ satisfying $\sigma\{\boldsymbol{T}\} = \{\boldsymbol{T}\}a_{\boldsymbol{\mu},\boldsymbol{l}}^{-j}$ are parameterized by marked $(\rho_\sigma,\boldsymbol{\gamma},\boldsymbol{l})$-tabloids on $\boldsymbol{\mu}$, where ρ_σ is the cycle type of σ.*

Corollary 3.3 and Corollary 3.4 below are obtained from Theorem 3.1 and Theorem 3.2 by applying $j = 1$.

Corollary 3.3. *For $\sigma \in S_m$ of cycle type ρ and an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$, it follows that*

$$\sum_{k \in \mathbb{Z}/l\mathbb{Z}} \zeta_l^k \, \mathrm{Char}\left(M^{\boldsymbol{\mu}}(k;\boldsymbol{l})\right)(\sigma) = |\{\, (Y,c) \,|\, \text{a marked } (\rho,\boldsymbol{l})\text{-tabloid on } \boldsymbol{\mu} \,\}|.$$

Corollary 3.4. *Let $\boldsymbol{\mu}$ be an $\boldsymbol{l}$-partition. Tabloids $\{\boldsymbol{T}\}$ on $\boldsymbol{\mu}$ satisfying $\sigma\{\boldsymbol{T}\} = \{\boldsymbol{T}\}a_{\boldsymbol{\mu},\boldsymbol{l}}^{-1}$ are parameterized by $\boldsymbol{l}$-fillings on $(\rho_\sigma, \boldsymbol{l})$-tabloids on $\boldsymbol{\mu}$, where ρ_σ is the cycle type of σ.*

The following corollary directly follows from Theorem 3.1.

Corollary 3.5. *For an integer j, let $\boldsymbol{\mu}$ be an $\boldsymbol{l}$-partition, $\boldsymbol{\gamma}$ an n-tuple of integers such that γ_h is the order of $\zeta_{l_h}^j$. For $\sigma \in S_m$ of cycle type ρ,*

$$\sum_{k \in \mathbb{Z}/l\mathbb{Z}} \zeta_l^{jk} \operatorname{Char}\left(M^{\boldsymbol{\mu}}(k; \boldsymbol{l})\right)(\sigma) = \sum_{k \in \mathbb{Z}/\gamma\mathbb{Z}} \zeta_\gamma^k \operatorname{Char}\left(M^{\boldsymbol{\mu}}(k; \boldsymbol{\gamma})\right)(\sigma)$$
$$= \left|\left\{ (Y, c) \,\middle|\, a \text{ marked } (\rho, \boldsymbol{\gamma})\text{-tabloid on } \boldsymbol{\mu} \right\}\right|.$$

Example 3.6. Let $\boldsymbol{\mu} = ((2,2),(4))$ and $\boldsymbol{l} = (2,1)$. First we consider the case where $j = 1$. In this case, all marked $((4,2,2),\boldsymbol{l})$-tabloids on $\boldsymbol{\mu}$ are the following:

It follows from Corollary 3.3 that

$$\sum_{k \in \mathbb{Z}/l\mathbb{Z}} \zeta_l^k \operatorname{Char}\left(M^{\boldsymbol{\mu}}(k; \boldsymbol{l})\right)((1234)(56)(78)) = 6.$$

Next consider the case where $j = 2$. Since $\zeta_{l_1} = \zeta_2 = -1$ and $\zeta_{l_2} = \zeta_1 = 1$, we have $\gamma_1 = \left|\langle \zeta_{l_1}^2 \rangle\right| = 1$ and $\gamma_2 = \left|\langle \zeta_{l_2}^2 \rangle\right| = 1$. All marked $((4,2,2),(1,1))$-tabloids on $\boldsymbol{\mu}$ are the following:

It follows from Theorem 3.1 that

$$\sum_{k \in \mathbb{Z}/l\mathbb{Z}} \zeta_l^{2k} \operatorname{Char}\left(M^{\boldsymbol{\mu}}(k; \boldsymbol{l})\right)((1234)(56)(78)) = 2.$$

4. Proof of Main Results

In this section, we prove Theorem 3.1 and Theorem 3.2. First we prove that Theorem 3.1 and Theorem 3.2 are equivalent; to prove the equivalence, we prepare Lemma 4.1. Next we prove Theorem 3.2 by giving an explicit parametrization; the correspondence φ defined in Definition 4.3 provides an explicit parametrization. We prove Theorem 3.2 first for the special element σ_ρ of the cycle type ρ, which is Lemma 4.7. We prepare Lemma 4.5 and Lemma 4.6 to prove Lemma 4.7. Last, in Theorem 4.8, we generalize Lemma 4.7 for general elements of the cycle type ρ. Theorem 4.8 is a realization of Theorem 3.2.

First we prove the equivalence of Theorem 3.1 and Theorem 3.2.

Lemma 4.1. *For an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ and $\sigma \in S_m$,*

$$\sum_{k \in \mathbb{Z}/l\mathbb{Z}} \zeta_l^{kj} \operatorname{Char}\left(M^{\boldsymbol{\mu}}(k; \boldsymbol{l})\right)(\sigma) = \left|\left\{ \{\boldsymbol{T}\} \in \mathbb{T}_{\boldsymbol{\mu}} \,\middle|\, \sigma\{\boldsymbol{T}\} = \{\boldsymbol{T}\}a_{\boldsymbol{\mu},\boldsymbol{l}}^{-j} \right\}\right|.$$

Proof. Let $\overline{\mathbb{T}_{\boldsymbol{\mu}}}$ be a subset of tabloids $\mathbb{T}_{\boldsymbol{\mu}}$ whose image is a basis of $M^{\boldsymbol{\mu}}(0;\boldsymbol{l})$. The image of $\overline{\mathbb{T}_{\boldsymbol{\mu}}}$ in $M^{\boldsymbol{\mu}}(k;\boldsymbol{l})$ is also a basis of $M^{\boldsymbol{\mu}}(k;\boldsymbol{l})$ for every k. Let $\mathbb{T}_{\boldsymbol{\mu}}^{(\sigma,j;\boldsymbol{l})}$ be a subset

$$\left\{\, \{\boldsymbol{T}\} \in \mathbb{T}_{\boldsymbol{\mu}} \,\middle|\, \sigma\{\boldsymbol{T}\} = \{\boldsymbol{T}\}a_{\boldsymbol{\mu},\boldsymbol{l}}^{j} \,\right\}$$

of $\mathbb{T}_{\boldsymbol{\mu}}$. We write $\overline{\mathbb{T}_{\boldsymbol{\mu}}^{(\sigma,j;\boldsymbol{l})}}$ for $\overline{\mathbb{T}_{\boldsymbol{\mu}}} \cap \mathbb{T}_{\boldsymbol{\mu}}^{(\sigma,j;\boldsymbol{l})}$.

A tabloid $\{\boldsymbol{T}\} \in \mathbb{T}_{\boldsymbol{\mu}}$ is mapped to a tabloid by $\sigma \in S_m$ in $M^{\boldsymbol{\mu}}$. Since $\widetilde{I}_{\boldsymbol{\mu}}(k;\boldsymbol{l})$ is generated by binomials $\{\boldsymbol{T}\}a_{\boldsymbol{\mu},\boldsymbol{l}}^{i} - \zeta_l^{ik}\{\boldsymbol{T}\}$ of tabloids, the representation matrix of $\sigma \in S_m$ for the basis $\overline{\mathbb{T}_{\boldsymbol{\mu}}}$ in $M^{\boldsymbol{\mu}}(k;\boldsymbol{l})$ is a matrix with entries in $\left\{\, 1, \zeta_l, \zeta_l^2, \ldots, \zeta_l^{l-1} \,\right\}$ whose nonzero elements appear exactly once for each row and for each column. Hence

$$\mathrm{Char}(M^{\boldsymbol{\mu}}(k;\boldsymbol{l}))(\sigma) = \sum_{j\in\mathbb{Z}/l\mathbb{Z}} \sum_{\boldsymbol{T}\in\mathbb{T}_{\boldsymbol{\mu}}^{(\sigma,j;\boldsymbol{l})}} \zeta_l^{kj} = \sum_{j\in\mathbb{Z}/l\mathbb{Z}} \left|\overline{\mathbb{T}_{\boldsymbol{\mu}}^{(\sigma,j;\boldsymbol{l})}}\right| \zeta_l^{kj}.$$

It follows from this equation that

$$\sum_{k\in\mathbb{Z}/l\mathbb{Z}} \zeta_l^{ik} \mathrm{Char}(M^{\boldsymbol{\mu}}(k;\boldsymbol{l}))(\sigma) = \sum_{k\in\mathbb{Z}/l\mathbb{Z}} \zeta_l^{ik} \sum_{j\in\mathbb{Z}/l\mathbb{Z}} \left|\overline{\mathbb{T}_{\boldsymbol{\mu}}^{(\sigma,j;\boldsymbol{l})}}\right| \zeta_l^{kj}$$

$$= \sum_{j\in\mathbb{Z}/l\mathbb{Z}} \sum_{k\in\mathbb{Z}/l\mathbb{Z}} \zeta_l^{k(i+j)} \left|\overline{\mathbb{T}_{\boldsymbol{\mu}}^{(\sigma,j;\boldsymbol{l})}}\right|.$$

Since $\sum_{k\in\mathbb{Z}/l\mathbb{Z}} \zeta_l^{kn} = 0$ for $n \neq 0$, this equation implies

$$\sum_{k\in\mathbb{Z}/l\mathbb{Z}} \zeta_l^{ik} \mathrm{Char}(M^{\boldsymbol{\mu}}(k;\boldsymbol{l}))(\sigma) = \sum_{k\in\mathbb{Z}/l\mathbb{Z}} \left|\overline{\mathbb{T}_{\boldsymbol{\mu}}^{(\sigma,-i;\boldsymbol{l})}}\right| = l\left|\overline{\mathbb{T}_{\boldsymbol{\mu}}^{(\sigma,-i;\boldsymbol{l})}}\right|.$$

Since $\left|\mathbb{T}_{\boldsymbol{\mu}}^{(\sigma,-i)}\right| = \left|\langle a_{\boldsymbol{\mu},\boldsymbol{l}}\rangle \overline{\mathbb{T}_{\boldsymbol{\mu}}^{(\sigma,-i;\boldsymbol{l})}}\right| = l\left|\overline{\mathbb{T}_{\boldsymbol{\mu}}^{(\sigma,-i;\boldsymbol{l})}}\right|$, we have

$$\sum_{k\in\mathbb{Z}/l\mathbb{Z}} \zeta_l^{ik} \mathrm{Char}(M^{\boldsymbol{\mu}}(k;\boldsymbol{l}))(\sigma) = \left|\mathbb{T}_{\boldsymbol{\mu}}^{(\sigma,-i;\boldsymbol{l})}\right|.$$

$\square$

It follows from Lemma 4.1 and the equation (1) that Theorem 3.1 and Theorem 3.2 are equivalent.

We construct a bijection between marked $(\rho_\sigma, \boldsymbol{\gamma}, \boldsymbol{l})$-tabloids on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ and tabloids $\{\boldsymbol{T}\}$ on $\boldsymbol{\mu}$ satisfying $\sigma\{\boldsymbol{T}\} = \{\boldsymbol{T}\}a_{\boldsymbol{\mu},\boldsymbol{l}}^{-1}$ to prove Theorem 3.2.

Definition 4.2. For a Young diagram $\rho \vdash m$, we define $n_{\rho,i}$, $N_{\rho,i}$, $\sigma_{\rho,i}$ and σ_ρ by the following:

$$n_{\rho,i} = 1 + \sum_{j=1}^{i-1} \rho_j,$$

$$N_{\rho,i} = \{\, n_{\rho,i}, n_{\rho,i}+1, \ldots, n_{\rho,i}+\rho_i-1 \,\} \subset \{\, 1, \ldots, m \,\},$$

$$\sigma_{\rho,i} = (n_{\rho,i}, n_{\rho,i}+1, \ldots, n_{\rho,i}+\rho_i-1) \in S_m,$$

$$\sigma_\rho = \sigma_{\rho,1}\sigma_{\rho,2}\sigma_{\rho,3}\cdots \in S_m.$$

For a Young diagram $\rho \vdash m$, it follows by definition that $\bigcup_i N_{\rho,i} = \{\, 1, \ldots, m \,\}$, $|N_{\rho,i}| = \rho_i$ and the cycle type of σ_ρ is ρ.

Definition 4.3. Let γ_h be the order of $a_{\mu^{(h)},l_h}^{j}$. For a marked $(\rho, \boldsymbol{\gamma}, \boldsymbol{l})$-tabloid (Y,c) on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$, $\{\varphi_j(Y,c)\}$ denotes the tabloid obtained by the following:

- Put the number $n_{\rho,i}$ on a box in the $c(i)$-th row of the inverse image $Y^{-1}(\{\, i \,\})$ for each i.

- Put the number $\sigma_\rho n$ on a box in the $(\overline{c-j}+ql_h)$-th row of $\mu^{(h)}$ if the number n is in the $(\overline{c}+ql_h)$-th row of $\mu^{(h)}$, where $\overline{c}$, $\overline{c-j} \in \mathbb{Z}/l_h\mathbb{Z} = \{1,\ldots,l_h\}$ and $q \in \mathbb{Z}$.

We define $\varphi_j(Y,c)$ to be the numbering sorted in ascending order in each row of $\{\varphi_j(Y,c)\}$.

Example 4.4. For a marked $((4,4,1),(2,1))$-tabloid

$$\left(\begin{array}{|c|c|} \hline 2^* & 2 \\ \hline 2 & 2 \\ \hline 1 & 1 \\ \hline 1^* & 1 \\ \hline \end{array} \, , \; \boxed{3^*} \right),$$

$$\varphi_1 \left(\begin{array}{|c|c|} \hline 2^* & 2 \\ \hline 2 & 2 \\ \hline 1 & 1 \\ \hline 1^* & 1 \\ \hline \end{array} \, , \; \boxed{3^*} \right) = \left(\begin{array}{|c|c|} \hline 5 & 7 \\ \hline 6 & 8 \\ \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array} \, , \; \boxed{9} \right).$$

For a marked $((4,4,1),(4,1),(2,1))$-tabloid

$$\left(\begin{array}{|c|c|} \hline 2^* & 2 \\ \hline 1 & 1 \\ \hline 2 & 2 \\ \hline 1^* & 1 \\ \hline \end{array} \, , \; \boxed{3^*} \right),$$

$$\varphi_2 \left(\begin{array}{|c|c|} \hline 2^* & 2 \\ \hline 1 & 1 \\ \hline 2 & 2 \\ \hline 1^* & 1 \\ \hline \end{array} \, , \; \boxed{3^*} \right) = \left(\begin{array}{|c|c|} \hline 5 & 7 \\ \hline 2 & 4 \\ \hline 6 & 8 \\ \hline 1 & 3 \\ \hline \end{array} \, , \; \boxed{9} \right).$$

Now we show that this correspondence φ_j provides a realization of Theorem 3.2.

Lemma 4.5. *For a marked $(\rho,\boldsymbol{\gamma},\boldsymbol{l})$-tabloid (Y,c) on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$, the tabloid $\{\varphi_j(Y,c)\}$ satisfies*

$$\sigma_\rho\{\varphi_j(Y,c)\} = \{\varphi_j(Y,c)\}a_{\boldsymbol{\mu},\boldsymbol{l}}^{-j},$$

where φ_j is the one defined in Definition 4.3 and γ_h is the order of $a_{\mu^{(h)},l_h}^{-j}$.

Proof. For the set $N_{\rho,i}$, σ_ρ acts as the cyclic permutation $\sigma_{\rho,i}$ of length ρ_i. Since the set $(\varphi_j(Y,c))^{-1}(N_{\rho,i})$ of boxes whose entries are in $N_{\rho,i}$ equals the inverse image $Y^{-1}(\{\,i\,\})$, we consider only $Y^{-1}(\{\,i\,\})$ now. Since we are allowed to change entries in the same row in $M^{\boldsymbol{\mu}}$, it follows from direct calculation that $\sigma_\rho(\{\varphi_j(Y,c)\})$ equals $\{\varphi_j(Y,c+1)\}$ over $Y^{-1}(\{\,i\,\})$, where $(c+1)(i) = c(i)+1$ in $\mathbb{Z}/\gamma_h\mathbb{Z}$. Since $\sigma_\rho\{\varphi_j(Y,c)\}$ equals $\{\varphi_j(Y,c+1)\}$ over $Y^{-1}(\{\,i\,\})$ for every i,

$$\sigma_\rho\{\varphi_j(Y,c)\} = \{\varphi_j(Y,c+1)\}.$$

On the other hand, $a_{\boldsymbol{\mu},\boldsymbol{l}}$ acts by

$$\{\varphi_j(Y,c)\}a_{\boldsymbol{\mu},\boldsymbol{l}}^{j} = \{\varphi_j(Y,c-1)\}.$$

Hence it follows that

$$\sigma_\rho\{\varphi_j(Y,c)\} = \{\varphi_j(Y,c+1)\} = \{\varphi_j(Y,c)\}a_{\boldsymbol{\mu},\boldsymbol{l}}^{-j}.$$

$\square$

Lemma 4.6. *Let a tabloid $\{\boldsymbol{T}\}$ on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ satisfy $\sigma_\rho\{\boldsymbol{T}\} = \{\boldsymbol{T}\}a_{\boldsymbol{\mu},\boldsymbol{l}}^{-j}$. If $\boldsymbol{T}^{-1}(n_{\rho,k})$ is a box in the $(\overline{r}+l_hq)$-th row of $\mu^{(h)}$, then $n \in N_{\rho,k}$ is in the $(\overline{r-(n-n_{\rho,k})j}+l_hq)$-th row of $\mu^{(h)}$, where $\overline{r}$ and $\overline{r-(n-n_{\rho,k})j} \in \mathbb{Z}/l_h\mathbb{Z} = \{\,1,\ldots,l_h\,\}$ and $q \in \mathbb{Z}$.*

Proof. Let $\boldsymbol{T}^{-1}(n_{\rho,k})$ be a box in the $(\overline{r} + l_h q)$-th row in $\mu^{(h)}$, where $\overline{r} \in \mathbb{Z}/l_h\mathbb{Z} = \{1, 2, \ldots, l_h\}$. Since $\sigma_\rho\{\boldsymbol{T}\} = \{\boldsymbol{T}\}a_{\boldsymbol{\mu},\boldsymbol{l}}^{-j}$, it follows, by the row where $n_{\rho,k}$ lies, that $\sigma_\rho n_{\rho,k}$ is in the $(\overline{r-j} + l_h q)$-th row in $\mu^{(h)}$, where $\overline{r-j} \in \mathbb{Z}/l_h\mathbb{Z} = \{1, 2, \ldots, l_h\}$. Similarly, we can specify boxes where elements of $N_{\rho,k}$ lie. Hence it follows that $n \in N_{\rho,k}$ lies in the $(\overline{r - (n - n_{\rho,k})j} + l_h q)$-th row, where $\overline{r - (n - n_{\rho,k})j} \in \mathbb{Z}/l_h\mathbb{Z} = \{1, \ldots, l_h\}$. $\square$

Lemma 4.7. *If γ_h is the order of $a_{\mu^{(h)},l_h}^j$, our correspondence φ_j provides a bijection between marked $(\rho, \boldsymbol{\gamma}, \boldsymbol{l})$-tabloids on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ and tabloids $\{\boldsymbol{T}\}$ on $\boldsymbol{\mu}$ satisfying $\sigma_\rho\{\boldsymbol{T}\} = \{\boldsymbol{T}\}a_{\boldsymbol{\mu},\boldsymbol{l}}^{-j}$.*

Proof. We construct the inverse map of φ_j. Let a tabloid $\{\boldsymbol{T}\}$ on $\boldsymbol{\mu}$ satisfy the equation $\sigma_\rho\{\boldsymbol{T}\} = \{\boldsymbol{T}\}a_{\boldsymbol{\mu},\boldsymbol{l}}^{-j}$. Let $\boldsymbol{T}$ be a numbering sorted in ascending order in each row of $\{\boldsymbol{T}\}$.

Let ψ be a map from $\{1, \ldots, m\}$ to $\mathbb{N}$ which maps $i \in N_{\rho,k}$ to k. For $i < i'$, $n \in N_{\rho,i}$ is smaller than $n' \in N_{\rho,i'}$. Since $\boldsymbol{T}$ is sorted in ascending order in each row, it follows from Lemma 4.6 that $\psi \circ \boldsymbol{T}$ is a $(\rho, \boldsymbol{\gamma}, \boldsymbol{l})$-tabloid.

Let $\psi_c(\boldsymbol{T})$ be a map from $\{i \,|\, \rho_i \neq 0\}$ to $\coprod_h \mathbb{Z}/l_h\mathbb{Z}$ defined by $\psi_c(i) = j \in \mathbb{Z}/\gamma_h\mathbb{Z}$ if $\boldsymbol{T}^{-1}(n_{\rho,i})$ is in $\mu^{(h)}$ and in the j-th row of the inverse image $\boldsymbol{T}^{-1}(N_{(\rho,i)})$. A pair $(\psi \circ \boldsymbol{T}, \psi_c \boldsymbol{T})$ is a marked $(\rho, \boldsymbol{\gamma}, \boldsymbol{l})$-tabloid on $\boldsymbol{\mu}$.

Since $n_{\rho,i}$ is in the same row in $\{\boldsymbol{T}\}$ as in $\{\varphi(\psi \circ \boldsymbol{T}, \psi_c \boldsymbol{T})\}$, it follows from Lemma 4.6 that

$$\{\varphi(\psi \circ \boldsymbol{T}, \psi_c(\boldsymbol{T}))\} = \{\boldsymbol{T}\}$$

and that

$$(\psi \circ \varphi(Y, c), \psi_c(\varphi(Y, c))) = (Y, c).$$

$\square$

Last we consider not only σ_ρ, but also general elements σ whose cycle type is ρ. We explicitly give parameterizations of Theorem 3.2 in the following theorem.

Theorem 4.8. *Let the cycle type of $\sigma \in S_m$ be ρ and let $\tau \in S_m$ satisfy $\tau\sigma_\rho\tau^{-1} = \sigma$. Then the set $\left\{\{\boldsymbol{T}\} \in \mathbb{T}_{\boldsymbol{\mu}} \,\middle|\, \sigma\{\boldsymbol{T}\} = \{\boldsymbol{T}\}a_{\boldsymbol{\mu},\boldsymbol{l}}^{-j}\right\}$ equals*

$$\{\{\tau\varphi_j(Y, c)\} \,|\, (Y, c) \text{ is a marked } (\rho, \boldsymbol{\gamma}, \boldsymbol{l})\text{-tabloid on } \boldsymbol{\mu}\}$$

for an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ of $\boldsymbol{m}$ and γ_h is the order of $a_{\mu^{(h)},l_h}^j$.

Proof. Since $\sigma = \tau\sigma_\rho\tau^{-1}$, the equation

$$\sigma\{\boldsymbol{T}\} = \{\boldsymbol{T}\}a_{\boldsymbol{\mu},\boldsymbol{l}}^{-j}$$

is equivalent to the equation

$$\sigma_\rho\tau^{-1}\{\boldsymbol{T}\} = \tau^{-1}\{\boldsymbol{T}\}a_{\boldsymbol{\mu},\boldsymbol{l}}^{-j}.$$

From Lemma 4.7, there exists a marked $(\rho, \boldsymbol{\gamma}, \boldsymbol{l})$-tabloid (Y, c) on $\boldsymbol{\mu}$ such that $\{\varphi_j(Y, c)\} = \tau^{-1}\{\boldsymbol{T}\}$. Hence this theorem follows. $\square$

Example 4.9. Let $\boldsymbol{\mu}$, $\boldsymbol{l}$ and ρ be the same as ones in Example 3.6, i.e., $\boldsymbol{\mu} = ((2, 2), 4)$, $\boldsymbol{l} = (2, 1)$ and $\rho = (4, 2, 2)$. First we consider the case where $j = 1$. In this case,

$$\left(\begin{array}{|c|c|} \hline 1^* & 1 \\ \hline 1 & 1 \\ \hline \end{array} \;,\; \boxed{2^*}\,\boxed{2}\,\boxed{3^*}\,\boxed{3} \right)$$

is a $(\rho, \boldsymbol{l})$-tabloid on $\boldsymbol{\mu}$. We have

$$\varphi_1\left(\begin{array}{|c|c|} \hline 1^* & 1 \\ \hline 1 & 1 \\ \hline \end{array} \;,\; \boxed{2^*}\,\boxed{2}\,\boxed{3^*}\,\boxed{3} \right) = \left(\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \;,\; \boxed{5}\,\boxed{6}\,\boxed{7}\,\boxed{8} \right).$$

Since $\sigma_\rho = (1,2,3,4)(5,6)(7,8)$ acts as

$$(1234)(56)(78)\left(\begin{array}{cc}1&3\\2&4\end{array}, \begin{array}{cccc}5&6&7&8\end{array}\right) = \left(\begin{array}{cc}2&4\\3&1\end{array}, \begin{array}{cccc}6&5&8&7\end{array}\right)$$

and

$$\left\{\left(\begin{array}{cc}2&4\\3&1\end{array}, \begin{array}{cccc}6&5&8&7\end{array}\right)\right\} = \left\{\left(\begin{array}{cc}2&4\\1&3\end{array}, \begin{array}{cccc}5&6&7&8\end{array}\right)\right\}$$

$$= \left\{\left(\begin{array}{cc}1&3\\2&4\end{array}, \begin{array}{cccc}5&6&7&8\end{array}\right)\right\}a_{\boldsymbol{\mu},\boldsymbol{l}}^{-1},$$

it follows that

$$\sigma_\rho\left\{\varphi_1\left(\begin{array}{cc}1^*&1\\1&1\end{array}, \begin{array}{ccc}2^*&2&3^*&3\end{array}\right)\right\} = \left\{\varphi_1\left(\begin{array}{cc}1^*&1\\1&1\end{array}, \begin{array}{ccc}2^*&2&3^*&3\end{array}\right)\right\}a_{\boldsymbol{\mu},\boldsymbol{l}}^{-1}.$$

Next we consider the case where $j = 2$. In this case,

$$\left(\begin{array}{cc}2^*&2\\3^*&3\end{array}, \begin{array}{cccc}1^*&1&1&1\end{array}\right)$$

is a $(\rho,(1,1),\boldsymbol{l})$-tabloid on $\boldsymbol{\mu}$. We have

$$\varphi_2\left(\begin{array}{cc}2^*&2\\3^*&3\end{array}, \begin{array}{cccc}1^*&1&1&1\end{array}\right) = \left(\begin{array}{cc}5&6\\7&8\end{array}, \begin{array}{cccc}1&2&3&4\end{array}\right).$$

Since σ_ρ acts as

$$(1234)(56)(78)\left\{\left(\begin{array}{cc}5&6\\7&8\end{array}, \begin{array}{cccc}1&2&3&4\end{array}\right)\right\} = \left\{\left(\begin{array}{cc}6&5\\8&7\end{array}, \begin{array}{cccc}2&3&4&1\end{array}\right)\right\}$$

$$= \left\{\left(\begin{array}{cc}5&6\\7&8\end{array}, \begin{array}{cccc}1&2&3&4\end{array}\right)\right\}$$

and $a_{\boldsymbol{\mu},\boldsymbol{l}}^{-2} = \varepsilon \in S_8$, it follows that

$$\sigma_\rho\left\{\varphi_1\left(\begin{array}{cc}2^*&2\\3^*&3\end{array}, \begin{array}{cccc}1^*&1&1&1\end{array}\right)\right\} = \left\{\varphi_1\left(\begin{array}{cc}2^*&2\\3^*&3\end{array}, \begin{array}{cccc}1^*&1&1&1\end{array}\right)\right\}a_{\boldsymbol{\mu},\boldsymbol{l}}^{-2}.$$

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